

ON THE SOLITON GEOMETRY IN MULTIDIMENSIONS

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Abstract

The connection between multidimensional soliton equations and differential geometry of surfaces/curves is discussed. A classes integrable surfaces (and/or curves) corresponding to the well known (2+1) - dimensional soliton equations are constructed. Examples of such equations include: the Davey - Stewartson, Zakharovs, Kadomtsev - Petviashvili, (2+1)-mKdV, (2+1)-KdV and Knizhnik - Novikov - Veselov equations. Using the presented geometrical formalism the equivalence between these soliton equations and their spin counterparts are established. The integrable curve/surface corresponding to the self-dual Yang-Mills equation is constructed.

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1 Introduction

Recently many works have been devoted to the study on motion of curves ad surfaces. One of the interesting motivations is the relation to soliton equations [1-10]. In particular, the geometrical formulation may set a suitable basis for the description of higher dimensional soliton equations [9-14]. In this note we would like to discuss the simplest and most transparent differential-geometric appearance of the some soliton equations in multidimensions (see, also , [16-27]).

Three equations of the differential geometry are may be starting points of the soliton geometry: the Serret-Frenet equation (SFE), the Gauss-Weingarten equation (GWE) and the Gauss-Mainardi-Codazzi equation (GMCE). We begin with the presentation of these equations.

1.1 The Serret-Frenet equation

The SFE has the form

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = \begin{pmatrix} 0 & k & 0 \\ -\beta k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \quad (1)$$

where k, τ are the curvature and torsion, x is the arc-length parameter, $\mathbf{e}_1, \mathbf{e}_2$ and \mathbf{e}_3 are the tangent, normal and binormal vectors, respectively with

$$\mathbf{e}_1^2 = \beta = \pm 1, \mathbf{e}_2^2 = \mathbf{e}_3^2 = 1 \quad (2)$$

where $\beta = +1$ corresponds to the focusing case and $\beta = -1$ corresponds to the defocusing case.

1.2 The Gauss-Weingarten equation

Let us consider a surface in R^3 equipped (locally) with coordinates x, y and defined by the position vector $\mathbf{r} = \mathbf{r}(x, y) \in R^3$. Then the GWE reads as

$$\mathbf{r}_{xx} = \Gamma_{11}^1 \mathbf{r}_x + \Gamma_{11}^2 \mathbf{r}_y + L \mathbf{n} \quad (3a)$$

$$\mathbf{r}_{xy} = \Gamma_{12}^1 \mathbf{r}_x + \Gamma_{12}^2 \mathbf{r}_y + M \mathbf{n} \quad (3b)$$

$$\mathbf{r}_{yy} = \Gamma_{22}^1 \mathbf{r}_x + \Gamma_{22}^2 \mathbf{r}_y + N \mathbf{n} \quad (3c)$$

$$\mathbf{n}_x = p_{11} \mathbf{r}_x + p_{12} \mathbf{r}_y \quad (3d)$$

$$\mathbf{n}_y = p_{21} \mathbf{r}_x + p_{22} \mathbf{r}_y \quad (3e)$$

where Γ_{ij}^k are the Christoffel symbols. This equation can be rewritten in the form

$$Z_x = AZ, \quad Z_y = BZ \quad (4)$$

where $Z = (\mathbf{r}_x, \mathbf{r}_y, \mathbf{n})^t$ and

$$A = \begin{pmatrix} \Gamma_{11}^1 & \Gamma_{11}^2 & L \\ \Gamma_{12}^1 & \Gamma_{12}^2 & M \\ p_{11} & p_{12} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \Gamma_{12}^1 & \Gamma_{12}^2 & M \\ \Gamma_{22}^1 & \Gamma_{22}^2 & N \\ p_{21} & p_{22} & 0 \end{pmatrix}. \quad (5)$$

On the other hand, in terms of the orthogonal frame

$$\mathbf{e}_1 = \frac{\mathbf{r}_x}{\sqrt{E}}, \quad \mathbf{e}_2 = \mathbf{n}, \quad \mathbf{e}_3 = \mathbf{e}_1 \wedge \mathbf{e}_2 \quad (6)$$

the GWE takes the form

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = A \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_y = B \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}. \quad (7)$$

Here

$$A = \frac{1}{\sqrt{E}} \begin{pmatrix} 0 & L & -\frac{g}{\sqrt{E}} \Gamma_{11}^2 \\ -\beta L & 0 & -gp_{12} \\ \frac{\beta g}{\sqrt{E}} \Gamma_{11}^2 & gp_{12} & 0 \end{pmatrix}, \quad B = \frac{1}{\sqrt{E}} \begin{pmatrix} 0 & M & -\frac{g}{\sqrt{E}} \Gamma_{12}^2 \\ -\beta M & 0 & -gp_{22} \\ \frac{\beta g}{\sqrt{E}} \Gamma_{12}^2 & gp_{22} & 0 \end{pmatrix} \quad (8)$$

where $g = EG - F^2$.

1.3 The Gauss-Mainardi-Codazzi equation

Integrability conditions for the GWE can be derived from

$$\mathbf{r}_{xxy} = \mathbf{r}_{xyx}, \quad \mathbf{r}_{xyy} = \mathbf{r}_{yyx} \quad (9a)$$

or

$$\mathbf{e}_{jxy} = \mathbf{e}_{jyx} \quad (9b)$$

and are equivalent to the following GMCE

$$A_y - B_x + [A, B] = 0 \quad (10)$$

where A, B are given by (5) or (8). In general, this equation is apparently nonintegrable. Only in some particular cases it is integrable (see, e.g. [33]). Also it admits some multidimensional extensions, e.g. in 2+1 dimensions. It is remarkable that some of these (2+1)-dimensional extensions are integrable, e.g. the M-LXII equation (19).

2 Soliton geometry in $d = 2$ dimensions

To make exposition self-contained, we expose here some necessary to us well known basis facts from the 2-dimensional soliton geometry. We must construct two-dimensional generalization of the SFE (1). The simplest and well known two-dimensional extension of the SFE has the form

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = A \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_y = B \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (11)$$

where

$$A = \begin{pmatrix} 0 & k & -\sigma \\ -\beta k & 0 & \tau \\ \sigma & -\tau & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & m_3 & -m_2 \\ -\beta m_3 & 0 & m_1 \\ \beta m_2 & -m_1 & 0 \end{pmatrix}. \quad (12)$$

In fact, this equation is equivalent to the GMCE (10) with

$$k = \frac{L}{\sqrt{E}}, \quad \sigma = \frac{g}{E} \Gamma_{11}^2, \quad \tau = -\frac{g}{\sqrt{E}} p_{12} \quad (13a)$$

$$m_1 = -\frac{g}{\sqrt{E}} p_{22}, \quad m_2 = \frac{g}{E} \Gamma_{12}^2, \quad m_3 = \frac{M}{\sqrt{E}} \quad (13b)$$

so that the compatibility condition of the equations (11) is the GMCE (10). In this paper we usually suppose that the variables k, τ, σ, m_j are some functions of λ (the spectral parameter).

3 Soliton geometry in $d = 3$ dimensions

3.1 Curves and Surfaces in 2+1 dimensions

It is known that the 2-dimensional SFE (11) or the GWE (7) admits several (2+1)-dimensional integrable and non integrable generalizations, which describe curves and surfaces in 2+1 dimensions. Here some of them.

3.1.1 The M-LIX equation

The M-LIX equation has the form [16]

$$\alpha \mathbf{e}_{1y} = f_1 \mathbf{e}_{1x} + \sum_{j=1}^n b_j \mathbf{e}_1 \wedge \frac{\partial^j}{\partial x^j} \mathbf{e}_1 + c_1 \mathbf{e}_2 + d_1 \mathbf{e}_3 \quad (14)$$

and the equations for $\mathbf{e}_{2y}, \mathbf{e}_{3y}, \mathbf{e}_{jt}$. The M-LIX equation (14) admits several integrable reductions (see, e.g. the subsection 3.3).

3.1.2 The M-LXVII equation

The M-LXVII equation looks like [16]

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = A(\lambda) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (15a)$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = B(\lambda) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_y + C(\lambda) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}. \quad (15b)$$

Here $A(\lambda), B(\lambda), C(\lambda)$ are some matrices. This equation also contents some integrable reductions (see, e.g. the subsection 3.5).

3.1.3 The M-LX equation

This extension of the GWE (7) or the SFE (1) has the form [16]

$$\alpha \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_y = A \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x + B \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (16a)$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = \sum_{j=0}^n C_j \frac{\partial^j}{\partial x^j} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (16b)$$

where A, B, C_j - some matrices. Some integrable reductions of the M-LX equation (16) were presented in [16].

3.1.4 The modified M-LXI equation

The modified M-LXI (mM-LXI) equation, which is the (2+1)-dimensional integrable extension of the GWE (7), usually we write in the form [16]

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = A \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_y = B \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = C \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (17)$$

with

$$A = \begin{pmatrix} 0 & k & -\sigma \\ -\beta k & 0 & \tau \\ \sigma & -\tau & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & m_3 & -m_2 \\ -\beta m_3 & 0 & m_1 \\ \beta m_2 & -m_1 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\beta \omega_3 & 0 & \omega_1 \\ \beta \omega_2 & -\omega_1 & 0 \end{pmatrix}. \quad (18)$$

In the case $\sigma = 0$ the mM-LXI equation transform to the M-LXI equation. We think that the M-LXI equation (17) is integrable in general and admits integrable reductions in particular (see, e.g. the subsection 3.2).

3.2 The mM-LXII equation and Soliton equations in 2+1 dimensions

Let us return to the mM-LXI equation (17). From (17) we obtain the following mM-LXII equation [16]

$$A_y - B_x + [A, B] = 0 \quad (19a)$$

$$A_t - C_x + [A, C] = 0 \quad (19b)$$

$$B_t - C_y + [B, C] = 0. \quad (19c)$$

The mM-LXI and mM-LXII equations are may be some of the (2+1)-dimensional simplest, interesting and important integrable extensions of the GWE (7)=(11) and GMCE (10), respectively. In fact, we observe that the M-LXII equation (19) is the particular case of the other integrable equation, namely the Bogomolny equation (BE). The BE has the form (see, e.g. [15])

$$\Phi_t + [\Phi, C] + A_y - B_x + [A, B] = 0 \quad (20a)$$

$$\Phi_y + [\Phi, B] + C_x - A_t + [C, A] = 0 \quad (20b)$$

$$\Phi_x + [\Phi, A] + B_t - C_y + [B, C] = 0. \quad (20c)$$

Hence as $\Phi = 0$ we obtain the mM-LXII (or M-LXII) equation (19). Hence follows and the other remarkable fact, namely, the mM-LXII equation is exact reduction of the Self-Dual Yang-Mills equation (SDYME)

$$F_{\alpha\beta} = 0, \quad F_{\bar{\alpha}\bar{\beta}} = 0, \quad F_{\alpha\bar{\alpha}} + F_{\beta\bar{\beta}} = 0 \quad (21)$$

Here

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} + [A_\mu, A_\nu] \quad (22)$$

and

$$\frac{\partial}{\partial x_\alpha} = \frac{\partial}{\partial z} - i \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x_{\bar{\alpha}}} = \frac{\partial}{\partial z} + i \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x_\beta} = \frac{\partial}{\partial x} - i \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial x_{\bar{\beta}}} = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}. \quad (23)$$

If in the SDYME (21) we take

$$A_\alpha = -iC, \quad A_{\bar{\alpha}} = iC, \quad A_\beta = A - iB, \quad A_{\bar{\beta}} = A + iB \quad (100)$$

and if A, B, C are independent of z , then the SDYME (21) reduces to the mM-LXII (or M-LXII as $\sigma = 0$) equation (19). As known that the LR of the SDYME has the form [28, 29]

$$(\partial_\alpha + \lambda \partial_{\bar{\beta}})\Psi = (A_\alpha + \lambda A_{\bar{\beta}})\Psi, \quad (\partial_\beta - \lambda \partial_{\bar{\alpha}})\Psi = (A_\beta - \lambda A_{\bar{\alpha}})\Psi \quad (24)$$

where λ is the spectral parameter satisfying the following set of the equations

$$\lambda_\beta = \lambda \lambda_{\bar{\alpha}}, \quad \lambda_\alpha = -\lambda \lambda_{\bar{\beta}}. \quad (25)$$

Apropos, the simplest solution of this set has may be the following form [16]

$$\lambda = \frac{a_1 x_{\bar{\alpha}} + a_2 x_{\bar{\beta}} + a_3}{a_2 x_\alpha - a_1 x_\beta + a_4}, \quad a_j = \text{consts.}$$

From (24) we obtain the LR of the M-LXII equation (19)

$$(-i\partial_t + \lambda \partial_{\bar{\beta}})\Psi = [-iC + \lambda(A + iB)]\Psi \quad (26a)$$

$$(\partial_\beta - i\lambda \partial_t)\Psi = [(A - iB) - i\lambda C]\Psi. \quad (26b)$$

So the mM-LXII equation (19) is integrable in the sense that it admits the LR (26). The other form LR for it we present in subsection 3.4 (see (62)).

Now we will establish the connection between the M-LXI and mM-LXII equations (17), (19) and soliton equations in 2+1 dimensions. Let us, we assume

$$\mathbf{e}_1 \equiv \mathbf{S}. \quad (27)$$

Moreover we introduce two complex functions q, p according to the following expressions

$$q = a_1 e^{ib_1}, \quad p = a_2 e^{ib_2} \quad (28)$$

where a_j, b_j are real functions. Now we ready to consider some examples.

3.2.1 The Ishimori and Davey-Stewartson equations

The Ishimori equation (IE) reads as

$$\mathbf{S}_t = \mathbf{S} \wedge (\mathbf{S}_{xx} + \alpha^2 \mathbf{S}_{yy}) + u_x \mathbf{S}_y + u_y \mathbf{S}_x \quad (29a)$$

$$u_{xx} - \alpha^2 u_{yy} = -2\alpha^2 \mathbf{S} \cdot (\mathbf{S}_x \wedge \mathbf{S}_y). \quad (29b)$$

In this case we have [21]

$$m_1 = \partial_x^{-1} [\tau_y - \frac{\epsilon}{2\alpha^2} M_2^{Ish} u], \quad m_2 = -\frac{1}{2\alpha^2 k} M_2^{Ish} u, \quad m_3 = \partial_x^{-1} [k_y + \frac{\tau}{2\alpha^2 k} M_2^{Ish} u] \quad (30)$$

and

$$\begin{aligned} \omega_1 &= \frac{1}{k} [-\omega_{2x} + \tau \omega_3], \quad \omega_2 = -k_x - \alpha^2 (m_{3y} + m_2 m_1) + i m_2 u_x \\ \omega_3 &= -k\tau + \alpha^2 (m_{2y} - m_3 m_1) + i k u_y + i m_3 u_x, \quad M_2^{Ish} = M_2|_{a=b=-\frac{1}{2}}. \end{aligned} \quad (31)$$

Functions q, p are given by (28) with

$$a_1^2 = a_1'^2 = \frac{1}{4}k^2 + \frac{|\alpha|^2}{4}(m_3^2 + m_2^2) - \frac{1}{2}\alpha_R km_3 - \frac{1}{2}\alpha_I km_2 \quad (32a)$$

$$b_1 = \partial_x^{-1} \left\{ -\frac{\gamma_1}{2ia_1'^2} - (\bar{A} - A + D - \bar{D}) \right\} \quad (32b)$$

$$a_2^2 = a_2'^2 = \frac{1}{4}k^2 + \frac{|\alpha|^2}{4}(m_3^2 + m_2^2) + \frac{1}{2}\alpha_R km_3 - \frac{1}{2}\alpha_I km_2 \quad (32c)$$

$$b_2 = \partial_x^{-1} \left\{ -\frac{\gamma_2}{2ia_2'^2} - (A - \bar{A} + \bar{D} - D) \right\} \quad (32d)$$

where

$$\begin{aligned} \gamma_1 = & i \left\{ \frac{1}{2}k^2\tau + \frac{|\alpha|^2}{2}(m_3 km_1 + m_2 k_y) - \right. \\ & \left. \frac{1}{2}\alpha_R(k^2 m_1 + m_3 k\tau + m_2 k_x) + \frac{1}{2}\alpha_I[k(2k_y - m_{3x}) - k_x m_3] \right\} \end{aligned} \quad (33a)$$

$$\begin{aligned} \gamma_2 = & -i \left\{ \frac{1}{2}k^2\tau + \frac{|\alpha|^2}{2}(m_3 km_1 + m_2 k_y) + \right. \\ & \left. \frac{1}{2}\alpha_R(k^2 m_1 + m_3 k\tau + m_2 k_x) + \frac{1}{2}\alpha_I[k(2k_y - m_{3x}) - k_x m_3] \right\}. \end{aligned} \quad (33b)$$

Here $\alpha = \alpha_R + i\alpha_I$. In this case, q, p satisfy the following DS equation

$$iq_t + q_{xx} + \alpha^2 q_{yy} + vq = 0 \quad (34a)$$

$$-ip_t + p_{xx} + \alpha^2 p_{yy} + vp = 0 \quad (34b)$$

$$v_{xx} - \alpha^2 v_{yy} + 2[(pq)_{xx} + \alpha^2 (pq)_{yy}] = 0. \quad (34c)$$

So we have proved that the IE (29) and the DS equation (34) are L-equivalent to each other. As well known that these equations are G-equivalent to each other [31]. Note that the IE contains two reductions: the Ishimori I equation as $\alpha_R = 1, \alpha_I = 0$ and the Ishimori II equation as $\alpha_R = 0, \alpha_I = 1$. The corresponding versions of the DS equation (34), we obtain as the corresponding values of the parameter α (for details, see, e.g. the ref. [21]).

3.2.2 The Zakharov II and M-IX equations

Now we find the connection between the Myrzakulov IX (M-IX) equation and the curves (the M-LXI equation). The M-IX equation reads as

$$\mathbf{S}_t = \mathbf{S} \wedge M_1 \mathbf{S} + A_2 \mathbf{S}_x + A_1 \mathbf{S}_y \quad (35a)$$

$$M_2 u = 2\alpha^2 \mathbf{S}(\mathbf{S}_x \wedge \mathbf{S}_y) \quad (35b)$$

where $\alpha, b, a = \text{const}$ and

$$M_1 = \alpha^2 \frac{\partial^2}{\partial y^2} + 4\alpha(b - a) \frac{\partial^2}{\partial x \partial y} + 4(a^2 - 2ab - b) \frac{\partial^2}{\partial x^2}$$

$$\begin{aligned}
M_2 &= \alpha^2 \frac{\partial^2}{\partial y^2} - 2\alpha(2a+1) \frac{\partial^2}{\partial x \partial y} + 4a(a+1) \frac{\partial^2}{\partial x^2} \\
A_1 &= i\{\alpha(2b+1)u_y - 2(2ab+a+b)u_x\} \\
A_2 &= i\{4\alpha^{-1}(2a^2b+a^2+2ab+b)u_x - 2(2ab+a+b)u_y\}. \tag{36}
\end{aligned}$$

The M-IX equation was introduced in [16] and is integrable. It admits several integrable reductions:

- i) the Ishimori equation (29) as $a = b = -\frac{1}{2}$;
- ii) the M-VIII equation as $a = b = -1, X = x/2, Y = y/\alpha, w = -\alpha^{-1}u_Y$

$$\mathbf{S}_t = \mathbf{S} \wedge \mathbf{S}_{YY} + w\mathbf{S}_Y \tag{35a}$$

$$w_X + w_Y + \mathbf{S}(\mathbf{S}_X \wedge \mathbf{S}_Y) = 0 \tag{35b}$$

- iii) the M-XXXIV equation as $a = b = -1, X = t$

$$\mathbf{S}_t = \mathbf{S} \wedge \mathbf{S}_{YY} + w\mathbf{S}_Y \tag{35a}$$

$$w_t + w_Y + \frac{1}{2}(\mathbf{S}_Y^2)_Y = 0 \tag{35b}$$

and so on [16]. In our case we have

$$m_1 = \partial_x^{-1}[\tau_y - \frac{\beta}{2\alpha^2}M_2u], \quad m_2 = -\frac{1}{2\alpha^2k}M_2u, \quad m_3 = \partial_x^{-1}[k_y + \frac{\tau}{2\alpha^2k}M_2u] \tag{37}$$

and

$$\omega_1 = \frac{1}{k}[-\omega_{2x} + \tau\omega_3], \tag{38a}$$

$$\omega_2 = -4(a^2 - 2ab - b)k_x - 4\alpha(b - a)k_y - \alpha^2(m_{3y} + m_2m_1) + m_2A_1 \tag{38b}$$

$$\omega_3 = -4(a^2 - 2ab - b)k\tau - 4\alpha(b - a)km_1 + \alpha^2(m_{2y} - m_3m_1) + kA_2 + m_3A_1. \tag{38c}$$

Functions q, p are given by (28) with

$$a_1^2 = \frac{|a|^2}{|b|^2}a_1'^2 = \frac{|a|^2}{|b|^2}\{(l+1)^2k^2 + \frac{|\alpha|^2}{4}(m_3^2 + m_2^2) - (l+1)\alpha_R km_3 - (l+1)\alpha_I km_2\} \tag{39a}$$

$$b_1 = \partial_x^{-1}\{-\frac{\gamma_1}{2ia_1'^2} - (\bar{A} - A + D - \bar{D})\} \tag{39b}$$

$$a_2^2 = \frac{|b|^2}{|a|^2}a_2'^2 = \frac{|b|^2}{|a|^2}\{l^2k^2 + \frac{|\alpha|^2}{4}(m_3^2 + m_2^2) - l\alpha_R km_3 + l\alpha_I km_2\} \tag{39c}$$

$$b_2 = \partial_x^{-1}\{-\frac{\gamma_2}{2ia_2'^2} - (A - \bar{A} + \bar{D} - D)\} \tag{39d}$$

where

$$\begin{aligned}
\gamma_1 &= i\{2(l+1)^2k^2\tau + \frac{|\alpha|^2}{2}(m_3km_1 + m_2k_y) - \\
&(l+1)\alpha_R[k^2m_1 + m_3k\tau + m_2k_x] + (l+1)\alpha_I[k(2k_y - m_{3x}) - k_xm_3]\} \tag{40a}
\end{aligned}$$

$$\gamma_2 = -i\{2l^2k^2\tau + \frac{|\alpha|^2}{2}(m_3km_1 + m_2k_y) -$$

$$l\alpha_R(k^2m_1 + m_3k\tau + m_2k_x) - l\alpha_I[k(2k_y - m_{3x}) - k_xm_3]\}. \quad (40b)$$

Here $\alpha = \alpha_R + i\alpha_I$. In this case, q, p satisfy the following Zakharov II (Z-II) equation [32]

$$iq_t + M_1q + vq = 0 \quad (41a)$$

$$ip_t - M_1p - vp = 0 \quad (41b)$$

$$M_2v = -2M_1(pq). \quad (41c)$$

As well known the Z-II equation admits several reductions: 1) the DS-I equation as $\alpha_R = 1, \alpha_I = 0$; 2) the DS-II equation as $\alpha_R = 0, \alpha_I = 1$; 3) the Z-III equation as $a = b = -1$ and so on [32, 16].

3.3 The M-LIX equation and Soliton equations in 2+1 dimensions

Now let us consider the connection between the M-LIX equation (14) and (2+1)-dimensional soliton equations. Mention that the M-LIX equation is one of (2+1)-dimensional extensions of the SFE (1). As example, let us consider the connection between the M-LIX equation and the Z-II equation (41). Let the spatial part of the M-LIX equation has the form [16]

$$\alpha \mathbf{e}_{1\eta} = i\mathbf{e}_1 \wedge \mathbf{e}_{1\xi} + i(q+p)\mathbf{e}_2 + (q-p)\mathbf{e}_3 \quad (42a)$$

and the equations for the $\mathbf{e}_{2y}, \mathbf{e}_{3y}$. This set of these equations can be considered as some generalization of the Belavin-Polyakov equation [28]

$$\mathbf{e}_{1\eta} = \pm \mathbf{e}_1 \wedge \mathbf{e}_{1\xi} \quad (43)$$

which arises in several physical applications. In terms of matrices the equations (42) we can write in the form [16]

$$\alpha \hat{e}_{1\eta} = \frac{1}{2}[\hat{e}_1, \hat{e}_{1\xi}] + i(q+p)\hat{e}_2 + (q-p)\hat{e}_3 \quad (44a)$$

$$\alpha \hat{e}_{2\eta} = [\hat{e}_1, \hat{e}_{2\xi}] + i\hat{e}_{3\xi} + i(A+B)\hat{e}_3 + (A-B)\hat{e}_2 - i(p+q)\hat{e}_1 \quad (44b)$$

$$\alpha \hat{e}_{3\eta} = [\hat{e}_1, \hat{e}_{3\xi}] - i\hat{e}_{2\xi} - i(A+B)\hat{e}_2 + (A-B)\hat{e}_3 + (p-q)\hat{e}_1 \quad (44c)$$

where

$$\hat{e}_1 = g^{-1}\sigma_3g, \quad \hat{e}_2 = g^{-1}\sigma_2g, \quad \hat{e}_3 = g^{-1}\sigma_1g, \quad \xi = \frac{y}{\alpha}, \quad \eta = 2x + \frac{2a+1}{\alpha}y. \quad (45)$$

Here σ_j are Pauli matrices. Hence follows that the matrix-function g satisfies the equations

$$\alpha g_y = B_1 g_x + B_0 g \quad (46)$$

with

$$B_0 = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} a+1 & 0 \\ 0 & a \end{pmatrix}. \quad (47)$$

To find the time evolution of matrices \hat{e}_j we require that the matrices \hat{e}_j satisfy the following set of the equations which is the time part of the M-LIX equation (14)

$$\begin{aligned} i\hat{e}_{1t} = & -\{[(2b+1)\hat{e}_1+1][\hat{e}_1, \hat{e}_{1\xi\xi}] + [2(2b+1)F^+ + F^-][\hat{e}_1, \hat{e}_{1\xi}] + Q\hat{e}_{1\xi}\} + \\ & -\{[(c_{12}+c_{21})+i(p-q)F^+]\hat{e}_2 - [i(c_{12}-c_{21})+(p-q)F^-]\hat{e}_3\} \quad (48a) \\ i\hat{e}_{2t} = & [-P+i(c_{11}-c_{22})]\hat{e}_3 + [i(p-q)F^+ + (c_{12}+c_{21})]\hat{e}_1 + -Q\hat{e}_3\hat{e}_{1\xi} + i\hat{e}_3\hat{e}_{1\xi\xi}\hat{e}_1 + i(p-q)\hat{e}_1\hat{e}_{1\xi} + \\ & \frac{i}{4}[i(2b+1)\hat{e}_2-\hat{e}_3][\hat{e}_1, \hat{e}_{1\xi\xi}] - T[\hat{e}_2, \hat{e}_{1\xi}] + [2b+1+\hat{e}_1]\{i\hat{e}_{1\xi\xi}\hat{e}_3 + \frac{1}{4}[\hat{e}_1, \hat{e}_{1\xi\xi}]\hat{e}_2 + \frac{1}{2}[\hat{e}_2, \hat{e}_{1\xi\xi}]\hat{e}_1\} \quad (48b) \end{aligned}$$

$$\begin{aligned} i\hat{e}_{3t} = & P\hat{e}_2 + [c_{11}-c_{22}-i(c_{12}-c_{21})-(p-q)F^-]\hat{e}_1 + Q\hat{e}_2\hat{e}_{1\xi} + T[\hat{e}_3, \hat{e}_{1\xi}] - \frac{1}{4}[(2b+1)\hat{e}_3-i\hat{e}_2][\hat{e}_1, \hat{e}_{1\xi\xi}] - \\ & i\hat{e}_2\hat{e}_{1\xi\xi}\hat{e}_1 - [2b+1+\hat{e}_1]\{i\hat{e}_{1\xi\xi}\hat{e}_2 + \frac{1}{4}[\hat{e}_1, \hat{e}_{1\xi\xi}]\hat{e}_3 + \frac{1}{2}[\hat{e}_3, \hat{e}_{1\xi\xi}]\hat{e}_1\} \quad (48c) \end{aligned}$$

where

$$Q = 2(2b+1)F^- + 4F^+ + (p+q), \quad P = i[2(2b+1)(F^-F^+ + F_\xi^+) + (p+q)F^+ + F^{-2} + F^{+2} + 2F_\xi^-],$$

$$T = 2(2b+1)F^+ + (2b+1)F^-\hat{e}_1 + 2F^+ + F^- + \frac{1}{2}(p+q)\hat{e}_1 + \frac{1}{2}(p-q)\hat{e}_3.$$

We note that as follows from these equations the vector \mathbf{e}_1 satisfies the M-IX equation

$$i\hat{e}_{1t} = \frac{1}{2}[\hat{e}_1, M_1\hat{e}_1] + A_1\hat{e}_{1y} + A_2\hat{e}_{1x} \quad (49a)$$

$$M_2u = \frac{\alpha^2}{2i}tr(\hat{e}_1([\hat{e}_{1x}, \hat{e}_{1y}])). \quad (49b)$$

The time part of the M-LIX equation (48) immediately gives the equation

$$g_t = 2C_2g_{xx} + C_1g_x + C_0g \quad (50)$$

where

$$C_0 = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \quad C_2 = \frac{2b+1}{2}I + \frac{1}{2}\sigma_3, \quad C_1 = iB_0. \quad (51)$$

Here

$$c_{12} = i(2b-a+1)q_x + i\alpha q_y, \quad c_{21} = i(a-2b)q_x - i\alpha p_y \quad (52)$$

and c_{jj} are the solutions of the following equations

$$(a+1)c_{11x} - \alpha c_{11y} = i[(2b-a+1)(pq)_x + \alpha(pq)_y] \quad (53a)$$

$$ac_{22x} - \alpha c_{22y} = i[(a-2b)(pq)_x - \alpha(pq)_y]. \quad (53b)$$

So we have identified the curve, given by the M-LIX equation (42) and (48) with the M-IX equation (35) \equiv (49). On the other hand, the compatibility condition of equations (46) and (50) is equivalent to the Z-II equation (41). So that we have also established the connection between the curve (the M-LIX equation) and the Z-II equation. And we have shown, once more that the M-IX equation (35) and the Z-II equation (41) are L-equivalent to each other. Finally we note as $a = b = -\frac{1}{2}$ from these results follow the corresponding connection between the M-LIX, Ishimori and DS equations [21]. And as $a = b = -1$ we get the relation between the M-VIII, M-LIX and Zakharov-III equations (for details, see [16]).

3.4 The M-LVIII, M-LXIII equations and Soliton equations in 2+1 dimensions

In the C-approach, our starting point is the following (2+1)-dimensional M-LVIII equation [16]

$$\mathbf{r}_t = \Upsilon_1 \mathbf{r}_x + \Upsilon_2 \mathbf{r}_y + \Upsilon_3 \mathbf{n} \quad (54a)$$

$$\mathbf{r}_{xx} = \Gamma_{11}^1 \mathbf{r}_x + \Gamma_{11}^2 \mathbf{r}_y + L \mathbf{n} \quad (54b)$$

$$\mathbf{r}_{xy} = \Gamma_{12}^1 \mathbf{r}_x + \Gamma_{12}^2 \mathbf{r}_y + M \mathbf{n} \quad (54c)$$

$$\mathbf{r}_{yy} = \Gamma_{22}^1 \mathbf{r}_x + \Gamma_{22}^2 \mathbf{r}_y + N \mathbf{n} \quad (54d)$$

$$\mathbf{n}_x = p_{11} \mathbf{r}_x + p_{12} \mathbf{r}_y \quad (54e)$$

$$\mathbf{n}_y = p_{21} \mathbf{r}_x + p_{22} \mathbf{r}_y. \quad (54f)$$

It is one of the (2+1)-dimensional generalizations of the GWE (3) and admits several integrable reductions. Practically, all integrable spin systems in 2+1 dimensions are some integrable reductions of the M-LVIII equation (54). For example, the isotropic M-I equation (76) or in the vector form

$$\mathbf{S}_t = (\mathbf{S} \wedge \mathbf{S}_y + u \mathbf{S})_x \quad (101a)$$

$$u_x = -\mathbf{S}(\mathbf{S}_x \wedge \mathbf{S}_y) \quad (101b)$$

is the particular case of the M-LVIII equation (54). In fact, let $\mathbf{r}_x = \mathbf{S}$, then the M-I equation (101) takes the form

$$\mathbf{r}_t = (u + \frac{MF}{\sqrt{g}}) \mathbf{r}_x - \frac{M}{\sqrt{g}} \mathbf{r}_y + \Gamma_{12}^2 \mathbf{n} \quad (102)$$

with

$$u = \partial_x^{-1} [\sqrt{g}(L\Gamma_{12}^2 - M\Gamma_{11}^2)]. \quad (103)$$

This equation (102) is in fact the particular case of the M-LVIII equation (54) with

$$\Upsilon_1 = u + \frac{MF}{\sqrt{g}}, \quad \Upsilon_2 = -\frac{M}{\sqrt{g}}, \quad \Upsilon_3 = \Gamma_{12}^2 \sqrt{g}. \quad (104)$$

Sometimes it is convenient to work using the B-approach. In this approach the starting equation is the following M-LXIII equation [16]

$$\mathbf{r}_{tx} = \Gamma_{01}^1 \mathbf{r}_x + \Gamma_{01}^2 \mathbf{r}_y + \Gamma_{01}^3 \mathbf{n} \quad (55a)$$

$$\mathbf{r}_{ty} = \Gamma_{02}^1 \mathbf{r}_x + \Gamma_{02}^2 \mathbf{r}_y + \Gamma_{02}^3 \mathbf{n} \quad (55b)$$

$$\mathbf{r}_{xx} = \Gamma_{11}^1 \mathbf{r}_x + \Gamma_{11}^2 \mathbf{r}_y + L \mathbf{n} \quad (55c)$$

$$\mathbf{r}_{xy} = \Gamma_{12}^1 \mathbf{r}_x + \Gamma_{12}^2 \mathbf{r}_y + M \mathbf{n} \quad (55d)$$

$$\mathbf{r}_{yy} = \Gamma_{22}^1 \mathbf{r}_x + \Gamma_{22}^2 \mathbf{r}_y + N \mathbf{n} \quad (55e)$$

$$\mathbf{n}_t = p_{01} \mathbf{r}_x + p_{02} \mathbf{r}_y \quad (55f)$$

$$\mathbf{n}_x = p_{11} \mathbf{r}_x + p_{12} \mathbf{r}_y \quad (55g)$$

$$\mathbf{n}_y = p_{21}\mathbf{r}_x + p_{22}\mathbf{r}_y. \quad (55h)$$

This equation follows from the M-LVIII equation (54) under the following conditions

$$\begin{aligned} \Gamma_{01}^1 &= \Upsilon_{1x} + \Upsilon_1\Gamma_{11}^1 + \Upsilon_2\Gamma_{12}^1 + \Upsilon_3p_{11} \\ \Gamma_{02}^1 &= \Upsilon_{2x} + \Upsilon_1\Gamma_{11}^2 + \Upsilon_2\Gamma_{12}^2 + \Upsilon_3p_{12} \\ \Gamma_{03}^1 &= \Upsilon_{3x} + \Upsilon_1L + \Upsilon_2M \\ p_{01} &= \frac{F\Gamma_{02}^3}{g}, \quad p_{02} = -\frac{E\Gamma_{02}^3}{g}, \quad g = EG - F^2 \end{aligned} \quad (56)$$

Note that the M-LXIII equation (69) usually we use in the following form

$$Z_x = AZ, \quad Z_y = BZ, \quad Z_t = CZ \quad (57)$$

where

$$A = \begin{pmatrix} \Gamma_{11}^1 & \Gamma_{11}^2 & L \\ \Gamma_{12}^1 & \Gamma_{12}^2 & M \\ p_{11} & p_{12} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \Gamma_{12}^1 & \Gamma_{12}^2 & M \\ \Gamma_{22}^1 & \Gamma_{22}^2 & N \\ p_{21} & p_{22} & 0 \end{pmatrix}, \quad C = \begin{pmatrix} \Gamma_{01}^1 & \Gamma_{01}^2 & \Gamma_{01}^3 \\ \Gamma_{02}^1 & \Gamma_{02}^2 & \Gamma_{02}^3 \\ \Gamma_{03}^1 & \Gamma_{03}^2 & 0 \end{pmatrix}. \quad (58)$$

The compatibility condition of the equations (57) gives

$$A_y - B_x + [A, B] = 0 \quad (59a)$$

$$A_t - C_x + [A, C] = 0 \quad (59b)$$

$$B_t - C_y + [B, C] = 0 \quad (59c)$$

that is the mM-LXII equation (19). These equations are equivalent the relations

$$\mathbf{r}_{yxx} = \mathbf{r}_{xyx}, \quad \mathbf{r}_{yyx} = \mathbf{r}_{xyy} \quad (60a)$$

$$\mathbf{r}_{txx} = \mathbf{r}_{xxt}, \quad \mathbf{r}_{txy} = \mathbf{r}_{xyt}, \quad \mathbf{r}_{tyy} = \mathbf{r}_{yyt}. \quad (60b)$$

Note that (60a) is the well known GMCE (10). So that the M-LXII equation (59) is one of the (2+1)-dimensional generalizations of the GMCE (10). In the orthogonal basis (6) the equation (57) takes the form

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = \frac{1}{\sqrt{E}} \begin{pmatrix} 0 & L & -\frac{g}{\sqrt{E}}\Gamma_{11}^2 \\ -\beta L & 0 & -gp_{12} \\ \frac{\beta g}{\sqrt{E}}\Gamma_{11}^2 & gp_{12} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (61a)$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_y = \frac{1}{\sqrt{E}} \begin{pmatrix} 0 & M & -\frac{g}{\sqrt{E}}\Gamma_{12}^2 \\ -\beta M & 0 & -gp_{22} \\ \frac{\beta g}{\sqrt{E}}\Gamma_{12}^2 & gp_{22} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (61b)$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = \frac{1}{\sqrt{E}} \begin{pmatrix} 0 & \Gamma_{01}^3 & -\frac{g}{\sqrt{E}}\Gamma_{01}^2 \\ -\beta\Gamma_{01}^3 & 0 & -g\Gamma_{03}^2 \\ \frac{\beta g}{\sqrt{E}}\Gamma_{01}^2 & g\Gamma_{03}^2 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}. \quad (61c)$$

This equations we can rewrite in terms of 2×2 matrices as

$$g_x = Ug, \quad g_y = Vg, \quad g_t = Wg \quad (62)$$

where

$$U = \frac{1}{2i\sqrt{E}} \begin{pmatrix} -\sqrt{g}p_{12} & L + i\sqrt{\frac{g}{E}}\Gamma_{11}^2 \\ L - i\sqrt{\frac{g}{E}}\Gamma_{11}^2 & \sqrt{g}p_{12} \end{pmatrix} \quad (63a)$$

$$V = \frac{1}{2i\sqrt{E}} \begin{pmatrix} -\sqrt{g}p_{22} & M - i\sqrt{\frac{g}{E}}\Gamma_{12}^2 \\ M + i\sqrt{\frac{g}{E}}\Gamma_{12}^2 & \sqrt{g}p_{22} \end{pmatrix} \quad (63b)$$

$$W = \frac{1}{2i\sqrt{E}} \begin{pmatrix} -\sqrt{g}\Gamma_{03}^2 & \Gamma_{01}^3 - i\sqrt{\frac{g}{E}}\Gamma_{01}^2 \\ \Gamma_{01}^3 + i\sqrt{\frac{g}{E}}\Gamma_{01}^2 & \sqrt{g}\Gamma_{03}^2 \end{pmatrix}. \quad (63c)$$

From these equations follow

$$U_y - V_x + [U, V] = 0 \quad (64a)$$

$$U_t - W_x + [U, W] = 0 \quad (64b)$$

$$V_t - W_y + [V, W] = 0 \quad (64c)$$

that is the other form of the mM-LXII equation (59). The equation (64a) is the GMCE (10). Note that the M-LXIII equation in the form (61) have the same form with the mM-LXI equation (17) with the following identifications

$$k = \frac{L}{\sqrt{E}}, \quad \sigma = \frac{g}{E}\Gamma_{11}^2, \quad \tau = -\frac{g}{\sqrt{E}}p_{12} \quad (65a)$$

$$m_1 = -\frac{g}{\sqrt{E}}p_{22}, \quad m_2 = \frac{g}{E}\Gamma_{12}^2, \quad m_3 = \frac{M}{\sqrt{E}} \quad (65b)$$

$$\omega_1 = -\frac{1}{\sqrt{E}}g\Gamma_{03}^2, \quad \omega_2 = \frac{g}{E}\Gamma_{03}^2, \quad \omega_3 = \frac{1}{\sqrt{E}}\Gamma_{01}^3. \quad (65c)$$

So that the set of the linear equations (62) can be considered as one of the form of the LR for the mM-LXII equation (19).

3.5 The M-LXVII equation and Soliton equations in 2+1 dimensions

In this section we consider curves and/or surfaces which are given by the following M-LXVII equation [16]

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_1} = B \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_2} = C \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_4} + D \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}. \quad (66)$$

Hence we have the 3-dimensional M-LXX equation

$$-bB_{\xi_4} + B_{\xi_2} - D_{\xi_1} + [B, D] = 0 \quad (67)$$

which is the compatibility condition of the equations (66) as $C = bI$.

3.5.1 The 3-dimensional Self-Dual Yang-Mills equation

To derive the SDYME in $d = 3$ dimensions we consider the following particular case of the M-LXVII equation [16]

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_1} = (A_1 - \lambda A_3) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (68a)$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_2} = \lambda \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_4} + (A_2 - \lambda A_4) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}. \quad (68b)$$

The compatibility condition of these equations yields the 3-dimensional SDYME

$$A_{2\xi_1} - A_{1\xi_2} + [A_2, A_1] = 0 \quad (69a)$$

$$-A_{3\xi_4} + [A_4, A_3] = 0 \quad (69b)$$

$$A_{1\xi_4} - A_{4\xi_1} + [A_1, A_4] = -A_{3\xi_2} + [A_2, A_3]. \quad (69c)$$

3.5.2 The Zakharov I equation

Now let us we consider the M-LXVII equation in the form [16]

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = (A_1 - \lambda A_3) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (70a)$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = \lambda \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_y + A_2 \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (70b)$$

with

$$A_1 = \begin{pmatrix} 0 & i(q-p) & (q+p) \\ -i(q-p) & 0 & 0 \\ -(q+p) & 0 & 0 \end{pmatrix} \quad (71a)$$

$$A_2 = \begin{pmatrix} 0 & (q+p)_y & i(p-q)_y \\ -(q+p)_y & 0 & v \\ -i(p-q)_y & -v & 0 \end{pmatrix} \quad (71b)$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \quad (71c)$$

The compatibility condition of these equations gives

$$iq_t = q_{xy} + vq \quad (72a)$$

$$-ip_t = p_{xy} + vp \quad (72b)$$

$$v_x = 2(pq)_y \quad (72c)$$

which for convenience we call the Zakharov I (Z-I) equation [32].

3.5.3 Integrable spin systems

Now we consider the case when the curves/surfaces are given by the following geometrical equation [16]

$$\begin{pmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{e}'_3 \end{pmatrix}_x = -\lambda A_3 \begin{pmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{e}'_3 \end{pmatrix} \quad (73a)$$

$$\begin{pmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{e}'_3 \end{pmatrix}_t = \lambda \begin{pmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{e}'_3 \end{pmatrix}_y - \lambda A_4 \begin{pmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{e}'_3 \end{pmatrix} \quad (73b)$$

with

$$A_3 = \begin{pmatrix} 0 & rS_1 & -irS_2 \\ -rS_1 & 0 & S_3 \\ irS_2 & -S_3 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & -ir[2iS_3S_{2y} - 2iS_2S_{3y} + iuS_1] & -r[2S_3S_{1y} - 2S_1S_{3y} - uS_2] \\ ir[2iS_3S_{2y} - 2iS_2S_{3y} + iuS_1] & 0 & -[ir^2(S^+S_y^- - S^-S_y^+) - uS_3] \\ r[2S_3S_{1y} - 2S_1S_{3y} - uS_2] & ir^2(S^+S_y^- - S^-S_y^+) - uS_3 & 0 \end{pmatrix}. \quad (74)$$

Here we have the additional condition

$$S_3^2 + r^2(S_1^2 + S_2^2) = 1, \quad r^2 = \pm 1. \quad (75)$$

The equation (73) with the condition (75) we call the M-LXVI equation. From the compatibility condition of the equations (73) we obtain the Myrzakulov I (M-I) equation

$$iS_t = ([S, S_y] + 2iuS)_x \quad (76a)$$

$$u_x = -\frac{1}{2i} \text{tr}(S[S_x, S_y]) \quad (76b)$$

where

$$S = \begin{pmatrix} S_3 & rS^- \\ rS^+ & -S_3 \end{pmatrix}, \quad S^\pm = S_1 \pm iS_2. \quad (77)$$

3.5.4 The (2+1)-dimensional mKdV equation

Let us now we consider the following version of the M-LXVII equation

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = (A_1 - \lambda A_3) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (78a)$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = \lambda^2 \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_y + (D_1\lambda + D_0) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (78b)$$

where A_1, A_3 are given by (71) and D_k are some matrices [16]. Then the complex functions q, p satisfy the (2+1)-dimensional complex mKdV equation

$$q_t + q_{xxy} - (qv_1)_x - v_2q = 0 \quad (79a)$$

$$p_t + p_{xxy} - (pv_1)_x - v_2p = 0 \quad (79b)$$

$$v_{1x} = 2(pq)_y \quad (79c)$$

$$v_{2x} = 2(pq_{xy} - p_{xy}q) \quad (79d)$$

which is the 2+1 dimensional complex mKdV [30]. If $p = \beta q$ is real, we get the following mKdV equation

$$q_t + q_{xxy} - (qv_1)_x = 0 \quad (80a)$$

$$v_{1x} = 2\beta(q^2)_y. \quad (80b)$$

3.5.5 The (2+1)-dimensional derivative NLSE

Let now we work with the following form of the M-LXVII equation [16]

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = (A_3\lambda^2 + A_1\lambda) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (81a)$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = \lambda^2 \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_y + (D_2\lambda^2 + D_1\lambda + D_0) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (81b)$$

where A_1, A_3 are given by (71). Then the complex functions q, p satisfy the Strachan equation [30]

$$iq_t = q_{xy} - 2ic(vq)_x \quad (82a)$$

$$-ip_t = p_{xy} + 2ic(vq)_x \quad (82b)$$

$$v_x = 2(pq)_y. \quad (82c)$$

3.5.6 The M-III_q equation

At last we consider the case

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = (A_3(c\lambda^2 + d\lambda) + A_1(2c\lambda + d)) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (83a)$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = 2(c\lambda^2 + d\lambda) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_y + (D_2\lambda^2 + D_1\lambda + D_0) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (83b)$$

where A_1, A_3 are given by (71). Then the complex functions q, p satisfy the (2+1)-dimensional M-III_q equation [16]

$$iq_t = q_{xy} - 2ic(vq)_x + d^2vq \quad (84a)$$

$$-ip_t = p_{xy} + 2ic(vq)_x + d^2vp \quad (84b)$$

$$v_x = 2(pq)_y. \quad (84c)$$

The M-III_q equation (84) admits two integrable reductions: the Strachan equation (82) as $d = 0$ and the Z-I equation (72) as $c = 0$.

4 Soliton geometry in $d = 4$ dimensions

There exist several equations of the soliton geometry in $d = 4$ dimensions. Some of them we present here.

4.1 The M-LXVIII equation

Consider the M-LXVIII equation [16]

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_1} = A \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_3} + B \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (85a)$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_2} = C \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_4} + D \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (85b)$$

where $\mathbf{e}_j^2 = 1, (\mathbf{e}_i \mathbf{e}_j) = \delta_{ij}$ and $A(\lambda), B(\lambda), C(\lambda), D(\lambda)$ are (3×3) -matrices, λ is some parameter, ξ_i are coordinates. This equation describes some four dimensional curves and/or "surfaces" in 3-dimensional space. It is one of main equations of the multidimensional soliton geometry and admits several integrable reductions.

4.2 The M-LXXI equation

Consider the M-LXXI equation [16]

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_1} = A \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_2} = B \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (86a)$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_4} = C \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_3} + D \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (86b)$$

The compatibility condition of these equations gives some nonlinear evolution equations (NEEs).

4.3 The M-LXI equation

Consider the 4-dimensional M-LXI equation [16]

$$\begin{aligned} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_1} &= A \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_2} = B \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \\ \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_3} &= C \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_4} = D \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}. \end{aligned} \quad (87)$$

The compatibility condition of these equations gives the following 4-dimensional M-LXII equation

$$A_{\xi_2} - B_{\xi_1} + [A, B] = 0, \quad A_{\xi_3} - C_{\xi_1} + [A, C] = 0, \quad A_{\xi_4} - D_{\xi_1} + [A, D] = 0 \quad (88a)$$

$$C_{\xi_2} - B_{\xi_3} + [C, B] = 0, \quad D_{\xi_2} - B_{\xi_4} + [D, B] = 0, \quad C_{\xi_4} - D_{\xi_3} + [C, D] = 0. \quad (88b)$$

This equation contents many interesting 4-dimnsional NEEs.

4.4 The M-LXX equation

From (85) we get the following M-LXX equation [16]

$$AD_{\xi_3} - CB_{\xi_4} + B_{\xi_2} - D_{\xi_1} + [B, D] = 0 \quad (89a)$$

$$A_{\xi_2} - CA_{\xi_4} + [A, D] = 0 \quad (89b)$$

$$[A, C] = 0 \quad (89c)$$

$$C_{\xi_1} - AC_{\xi_3} + [C, B] = 0. \quad (89d)$$

If we choose

$$A = aI, \quad C = bI, \quad a, b = \text{consts} \quad (90)$$

then the M-LXX equation (89) takes the form

$$aD_{\xi_3} - bB_{\xi_4} + B_{\xi_2} - D_{\xi_1} + [B, D] = 0. \quad (91)$$

4.5 The SDYME

Now we assume that

$$B = A_1 - \lambda A_3, \quad D = A_2 - \lambda A_4, \quad a = b = \lambda. \quad (92)$$

So that the M-LXVIII equation takes the form [16]

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_1} = \lambda \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_3} + (A_1 - \lambda A_3) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad (93a)$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_2} = \lambda \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_{\xi_4} + (A_2 - \lambda A_4) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}. \quad (93b)$$

From (91) we obtain the SDYME

$$A_{2\xi_1} - A_{1\xi_2} + [A_2, A_1] = 0 \quad (94a)$$

$$A_{4\xi_3} - A_{3\xi_4} + [A_4, A_3] = 0 \quad (94b)$$

$$A_{1\xi_4} - A_{4\xi_1} + [A_1, A_4] = A_{2\xi_3} - A_{3\xi_2} + [A_2, A_3] \quad (94c)$$

or

$$F_{\xi_1\xi_2} = 0, \quad F_{\xi_3\xi_4} = 0, \quad F_{\xi_4\xi_1} - F_{\xi_3\xi_2} = 0. \quad (94d)$$

Here

$$F_{\xi_i\xi_k} = A_{k\xi_i} - A_{i\xi_k} + [A_k, A_i].$$

The SDYME (94) on a connection A are the self-duality conditions on the curvature under the Hodge star operation

$$F = *F \quad (95a)$$

or in index notation

$$F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\delta} F^{\rho\delta} \quad (95b)$$

where $*$ is Hodge operator, $\epsilon_{\mu\nu\rho\delta}$ stands for the completely antisymmetric tensor in four dimensions with the convention: $\epsilon_{1234} = 1$. The SDYME is integrable by the Inverse Scattering Transform method (see, e.g. [28,29]). The Lax representation (LR) of the SDYME has the form [28, 29]

$$\Phi_{\xi_1} - \lambda \Phi_{\xi_3} = (A_1 - \lambda A_3) \Phi \quad (96a)$$

$$\Phi_{\xi_2} - \lambda \Phi_{\xi_4} = (A_2 - \lambda A_4) \Phi. \quad (96b)$$

Hence follows that for the SDYME the spectral parameter λ satisfies the equations

$$\lambda_{\xi_1} = \lambda \lambda_{\xi_3}, \quad \lambda_{\xi_2} = \lambda \lambda_{\xi_4}. \quad (97)$$

These equations have the following solutions

$$\lambda = \frac{n_1 \xi_3 + n_3}{n_4 - n_1 \xi_1}, \quad \lambda = \frac{m_1 \xi_4 + m_3}{m_4 - m_1 \xi_2}. \quad (98)$$

So that the general solution of the set (97) has the form

$$\lambda = \frac{n_1 \xi_3 + n_3 + m_1 \xi_4}{n_4 - n_1 \xi_1 - m_1 \xi_2} \quad (99)$$

where $m_i, n_i = \text{constants}$. The corresponding solution of the SDYME (94) is called the breaking (overlapping) solutions.

5 Conclusion

In this note, we have formulated the some classes of the motion of curves/surfaces in $d = 3, 4$ -dimensional space using the differential geometry. It is shown that some of these curves/surfaces are integrable in the sense that they are connected with the well known integrable equations in multidimensions. Examples include practically all known multidimensional integrable (soliton) equations: the DS, Zakharovs, NLS-types, (2+1)-KdV, mKdV, Ishimori, M-IX, M-I equations and the SDYME. In particular, we have shown that one of (2+1)-dimensional extensions of the GMCE (10), namely, **the M-LXII (or mM-LXII) equation (19)=(59) is integrable** as the particular case of the integrable BE (20). It means that **the M-LXII equation is exact reduction of the famous SDYME**. In turn, it indicate that **almost all known soliton eqiations in 2+1 dimensions are exact reductions of the SDYME** since these equations obtained from the mM-LXII (or M-LXII) equation as some particular cases (see for instance refs [? - ?]).

Although main elements of our approach have been established, but there remain many problems to be studied. The study of some of these problems will be the subjects of our future works. Here only we note that there exists the other approach to study of the multidimensional soliton geometry developing mainly by Konopelchenko and coworkers (see, e.g. refs. [9, 11-14] and references therein). The main tool of this approach is a generalized Weierstrass representation for a conformal immersion of surfaces into R^3 or R^4 , and also a linear problem related with this representation. A consideration of the linear problem along with the Weierstrass representation allows to express integrable deformations of surfaces via such hierarchies soliton equations as a Veselov-Novikov hierarchy, DS hierarchy and so on. Thus, in this context, this and our approaches, developing in parallel, have the common purpose, namely, the construction surfaces (and curves) inducing by multidimensional soliton equations and complement to each other.

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